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ASYMPTOTIC DISTRIBUTIONS OF THE SAMPLE MEAN
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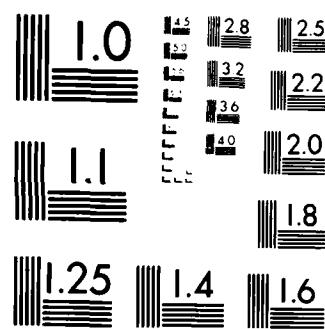
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ASYMPTOTIC DISTRIBUTIONS OF THE
SAMPLE MEAN, AUTOCOVARIANCES AND
AUTOCORRELATIONS OF LONG-MEMORY
TIME SERIES

J. R. M. Hosking

**Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705**

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ABSTRACT

We derive the asymptotic distributions of the sample mean, autocovariances and autocorrelations for a time series whose autocovariance function γ_k has the power-law decay $\gamma_k \sim \lambda k^{-\alpha}$, $\lambda > 0$, $0 < \alpha < 1$, as $k \rightarrow \infty$. The results differ in many respects from the corresponding results for short-memory processes, whose autocovariance functions are absolutely summable. For long-memory processes the variances of the sample mean, and of the sample autocovariances and autocorrelations for $0 < \alpha < 1/2$, are not of order n^{-1} asymptotically. When $0 < \alpha < 1/2$ the asymptotic distribution of the sample autocovariances and autocorrelations is not Normal.

AMS (MOS) Subject Classifications: 62E20, 62M10

Key Words: ARIMA process, asymptotic distribution theory, autocovariance, autocorrelation, fractional differencing, long memory, mean, time series

Work Unit No. 4 (Statistics and Probability)

*Institute of Hydrology, Maclean Building, Crowmarsh Gifford, Wallingford, Oxon OX10 8BB, England.

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SIGNIFICANCE AND EXPLANATION

Persistence, or long memory, is the presence in a time series of significant dependence between observations a long time span apart. Correct identification of long memory in an observed time series can greatly improve the accuracy of long-range forecasts of the series, and can give a better understanding of the physical processes which generate the observed series. The long-memory phenomenon has been observed by researchers in a number of areas of application including economics, geophysics, hydrology and meteorology.

Identification of long memory in an observed time series y_1, \dots, y_n is often based on the failure of the sample autocorrelations

$$r_k = c_k/c_0, \quad c_k = n^{-1} \sum_{t=1}^{n-k} (y_t - \bar{y})(y_{t+k} - \bar{y}), \quad k = 0, 1, 2, \dots, n-1,$$

where $\bar{y} = (y_1 + \dots + y_n)/n$ is the sample mean, to die away rapidly to zero as k increases. Because r_k is a random quantity its probability distribution must be known before accurate inferences may be drawn concerning it. This report derives the distributions of the sample autocorrelations and related quantities when the sample size n is large, thereby facilitating the diagnosis of long memory in an observed time series.

ASYMPTOTIC DISTRIBUTIONS OF THE SAMPLE MEAN, AUTOCOVARIANCES
AND AUTOCORRELATIONS OF LONG-MEMORY TIME SERIES

J. R. M. Hosking*

1. INTRODUCTION

Let $\{y_t : t \in \mathbb{Z}\}$ be a second-order stationary time series with mean $Ey_t = \mu$ and autocovariance function $\gamma_k = E\{(y_t - \mu)(y_{t+k} - \mu)\}$. We say that $\{y_t\}$ has short memory or long memory according as to whether $\sum_{k=-\infty}^{\infty} |\gamma_k|$ is convergent or divergent. Most of the theory and practice of the analysis of stationary time series is concerned with short-memory series, but the use of long-memory models has been considered by a number of authors, for example Mandelbrot and Wallis (1969), Granger (1980), Granger and Joyeux (1980), Hosking (1981, 1984), Jonas (1981), Janacek (1982), Geweke and Porter-Hudak (1983) and Li and McLeod (1984). The models considered by these authors mostly have autocovariance functions which satisfy

$$\gamma_k \sim \lambda k^{-\alpha}, \quad \lambda > 0, \quad 0 < \alpha < 1, \quad \text{as } k \rightarrow \infty. \quad (1)$$

In this report we consider the large-sample properties - mean, variance and asymptotic distribution - of the sample mean, autocovariances and autocorrelations of long-memory time series. For a time series with an autocovariance function of the form (1) and a Normal marginal distribution, our results are complete and are summarized in Table 1.

Most of the results in Table 1 - perhaps all of them - are valid under wider conditions than Normality of the time series. In our proofs we shall typically assume that $\{y_t\}$ has the representation

$$y_t = \mu + \sum_{j=0}^{\infty} \psi_j a_{t-j} \quad (2)$$

where

$$\psi_j \sim v_j^{-\beta}, \quad v > 0, \quad \frac{1}{2} < \beta < 1, \quad \text{as } j \rightarrow \infty \quad (3)$$

*Institute of Hydrology, Maclean Building, Crowmarsh Gifford, Wallingford, Oxon OX10 8BB, England

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TABLE 1. Asymptotic bias, variance and distribution of sample statistics for a Normal time series with autocovariance function (1): n is sample size.

Statistic	Range of α	Order of magnitude of: bias variance		Asymptotic Distribution
mean	$0 < \alpha < 1$	0	$n^{-\alpha}$	Normal
autocovariances and autocorrelations	$0 < \alpha < \frac{1}{2}$	$n^{-\alpha}$	$n^{-2\alpha}$	"modified Rosenblatt": cumulants defined by (13)-(16) below
	$\alpha = \frac{1}{2}$	$n^{-\alpha}$	$n^{-1} \log n$	
	$\frac{1}{2} < \alpha < 1$	$n^{-\alpha}$	n^{-1}	Normal

and $\{a_t : t \in \mathbb{Z}\}$ is a white-noise process consisting of independent and identically distributed random variables whose distribution we shall require to satisfy any of the following conditions:

$$Ea_t^2 = \sigma^2 < \infty; \quad (4a)$$

$$Ea_t^4 = \sigma^4(3 + \kappa) < \infty; \quad (4b)$$

$$Ea_t^m < \infty \text{ for all positive integers } m; \quad (4c)$$

$$a_t \sim N(0, \sigma^2). \quad (4d)$$

Some of our assumptions are unnecessarily restrictive but are imposed for convenience of presentation; some relaxations of them are considered in Section 7. When (2) and (4a) hold, (1) is a consequence of (3), as we now demonstrate.

Lemma 1.

Suppose that the time series $\{y_t\}$ satisfies (2), (3) and (4a). Then the autocovariance function γ_k of $\{y_t\}$ satisfies (1), with $\alpha = 2\beta - 1$ and $\lambda = \sigma^2 v^2 \Gamma(2\beta-1)\Gamma(1-\beta)/\Gamma(\beta)$.

Proof. Let $\psi_j = v\Gamma(j+1-\beta)/\Gamma(j+1)$, $j = 0, 1, 2, \dots$. Then $\{\psi_j\}$ is a bounded positive decreasing sequence and $\psi_j \sim vj^{-\beta} \sim \psi_0$ as $j \rightarrow \infty$. Thus

(i) there exists $C > 0$ s.t. $|\psi_j| < C\psi_0$, $j = 0, 1, 2, \dots$;

(ii) $\forall \epsilon > 0$ there exists J s.t. $j \geq J \Rightarrow (1 - \epsilon)\psi_0 < \psi_j < (1 + \epsilon)\psi_0$.

From (2) and (4a) we have $\gamma_k = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+k}$. Now

$$\begin{aligned}
 \Gamma_k &\equiv \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+k} = \sigma^2 \nu^2 \frac{\Gamma(1-\beta)\Gamma(k+1-\beta)}{\Gamma(k+1)} \sum_{j=0}^{\infty} \frac{1}{\Gamma(j+1)} \frac{\Gamma(j+1-\beta)}{\Gamma(1-\beta)} \frac{\Gamma(j+k+1-\beta)}{\Gamma(k+1-\beta)} \frac{\Gamma(k+1)}{\Gamma(j+k+1)} \\
 &= \sigma^2 \nu^2 \frac{\Gamma(1-\beta)\Gamma(k+1-\beta)}{\Gamma(k+1)} F(1-\beta, k+1-\beta; k+1; 1) \\
 &= \sigma^2 \nu^2 \frac{\Gamma(2\beta-1)\Gamma(1-\beta)}{\Gamma(\beta)} \frac{\Gamma(k+1-\beta)}{\Gamma(k+\beta)} \\
 &\sim \sigma^2 \nu^2 \frac{\Gamma(2\beta-1)\Gamma(1-\beta)}{\Gamma(\beta)} k^{2\beta-1} \text{ as } k \rightarrow \infty;
 \end{aligned}$$

here $F(a, b; \gamma, x)$ is the hypergeometric function. Thus for all $\epsilon > 0$ we have

$$\begin{aligned}
 |k^{2\beta-1}(\gamma_k - \Gamma_k)| &\leq \sigma^2 k^{2\beta-1} \left\{ \left| \sum_{j=0}^{J-1} \psi_j \psi_{j+k} \right| + \left| \sum_{j=0}^{J-1} \psi_j \psi_{j+k} \right| + \left| \sum_{j=J}^{\infty} \psi_j \psi_{j+k} - \sum_{j=J}^{\infty} \psi_j \psi_{j+k} \right| \right\} \\
 &\leq \sigma^2 k^{2\beta-1} J(\nu^2 + 1) \psi_0 \psi_k + \sigma^2 k^{2\beta-1} (2\epsilon + \epsilon^2) \sum_{j=J}^{\infty} \psi_j \psi_{j+k}.
 \end{aligned}$$

This expression can be bounded by a constant multiple of ϵ for k sufficiently large, since the first term is $O(k^{2\beta-1} \psi_k) = O(k^{\beta-1}) \rightarrow 0$ as $k \rightarrow \infty$ and the second term is bounded by $(2\epsilon + \epsilon^2) k^{2\beta-1} \Gamma_k$ with $k^{2\beta-1} \Gamma_k$ being bounded in k . Thus $\gamma_k \sim \Gamma_k$ as $k \rightarrow \infty$ and the result follows.

The converse of Lemma 1 is not true. For example if (2) and (4a) hold and $\psi_j = 0$ if $j = 2^m$ for some positive integer m , $\psi_j = j^{-\beta}$ otherwise, then it is easy to show that (1) is still true.

An outline of this report is as follows. In Sections 2, 3 and 4 we derive the asymptotic properties of the sample mean, autocovariances and autocorrelations respectively. The cumulants of two non-Normal distributions arising from the asymptotic theory of Sections 3 and 4 are investigated in Section 5. In Section 6 we apply our results to the family of fractionally differenced ARIMA(p,d,q) processes, a particularly widely known class of long-memory time series. Section 7 contains some indications of possible extensions of our results.

2. THE SAMPLE MEAN

The sample mean of a realization $\{y_t : t = 1, \dots, n\}$ of a time series is

$$\bar{y} = n^{-1} \sum_{t=1}^n y_t$$

and has mean u and variance

$$\text{var } \bar{y} = n^{-2} \sum_{t=1}^n \sum_{u=1}^n \gamma_{t-u} = n^{-2} \left\{ \sum_{t=1}^n (n-t)\gamma_t + n\gamma_0 \right\}. \quad (5)$$

Lemma 2

Let $\{y_t : t = 1, \dots, n\}$ be a sample from a second-order stationary time series whose covariance function γ_k satisfies (1). Then

$$\text{var } \bar{y} \sim \frac{2\lambda n^{-\alpha}}{(1-\alpha)(2-\alpha)} \text{ as } n \rightarrow \infty.$$

Proof. Write (5) as

$$\text{var } \bar{y} = 2n^{-\alpha} \cdot n^{-1} \sum_{t=1}^n (1 - \frac{t}{n}) n^\alpha \gamma_t + n^{-1} \gamma_0.$$

As $n \rightarrow \infty$ the sum $n^{-1} \sum (1 - t/n) n^\alpha \gamma_t$ converges to the integral

$$\int_0^1 (1-t)\lambda t^{-\alpha} dt = 1/(1-\alpha)(2-\alpha).$$

Since $n^{-1} \gamma_0 = o(n^{-\alpha})$ as $n \rightarrow \infty$ the result follows.

Theorem 1.

Suppose that the time series $\{y_t\}$ satisfies (1)-(3) and (4a). Then

$$n^{\alpha/2} (\bar{y} - u) \xrightarrow{w} N(0, 2\lambda/(1-\alpha)(2-\alpha)). \text{ as } n \rightarrow \infty.$$

Proof. Since $\gamma_j = o(j^{-(1+\alpha)/2})$ with $\alpha > 0$, we have $\sum_j \gamma_j^2 < \infty$. Furthermore, from Lemma 2,

$$E\left(\sum_{t=1}^n y_t\right)^2 \sim \frac{2\lambda n^{2-\alpha}}{(1-\alpha)(2-\alpha)} + \infty \text{ as } n \rightarrow \infty.$$

The result now follows from Theorem 18.6.5 of Ibragimov and Linnik (1971).

Note. The proof of their Theorem 18.6.5 given by Ibragimov and Linnik (1971) is defective, but can easily be corrected, as indicated in the Appendix to this report.

Remark. Theorem 1 is also valid if $\{y_t\}$ satisfies (2) and (4a) and has an autocovariance function γ_k with $\gamma_0 < \infty$, $\gamma_k \sim \lambda k^{-\alpha}$, $\lambda < 0$, $1 < \alpha < 2$. The proof is unaltered.

3. SAMPLE AUTOCOVARIANCES

We define the sample autocovariances of a realization $\{y_t : t = 1, \dots, n\}$ of a time series by

$$c_k = n^{-1} \sum_{t=1}^{n-k} (y_t - \bar{y})(y_{t+k} - \bar{y}), \quad k = 0, 1, \dots, n-1.$$

Other definitions of sample autocovariance have been used by some authors, but the more common variants, for example those defined by Anderson (1971, chapter 8), differ from c_k by quantities of stochastic order $O_p(n^{-1})$ and have asymptotic properties identical to those of c_k . We also define

$$\tilde{c}_k = n^{-1} \sum_{t=1}^{n-k} (y_t - \mu)(y_{t+k} - \mu), \quad k = 0, 1, \dots, n-1;$$

these quantities are estimators of γ_k when the sample mean is known. For short-memory processes the asymptotic distributions of c_k and \tilde{c}_k are identical. For long-memory processes with autocovariance function (1) this is only true if $1/2 < \alpha < 1$. When $0 < \alpha < 1/2$, replacement of μ by \bar{y} has an effect which is not negligible even in large samples, for it introduces a bias into the estimated autocovariances which is of the same order of magnitude as their standard deviation.

Theorem 2.

Let $\{y_t\}$ be a stationary time series satisfying (1). Then the asymptotic bias of c_k is given by

$$E c_k - \gamma_k \sim \frac{-2\lambda n^{-\alpha}}{(1-\alpha)(2-\alpha)} \quad \text{as } n \rightarrow \infty. \quad (6)$$

If in addition $\{y_t\}$ satisfies (2), (3) and (4b), then the asymptotic covariance of the c_k is given by

$$\text{cov}(c_k, c_l) \sim \begin{cases} 2\lambda^2 K_2 n^{-2\alpha} & \text{if } 0 < \alpha < 1/2, \\ 4\lambda^2 n^{-1} \log n & \text{if } \alpha = 1/2, \end{cases} \quad (7a)$$

$$\text{cov}(c_k, c_l) \sim \begin{cases} n^{-1} \left\{ \sum_{s=-\infty}^{\infty} (\gamma_s \gamma_{s+k-l} + \gamma_{s+k} \gamma_{s-l}) + K \gamma_k \gamma_l \right\} & \text{if } 1/2 < \alpha < 1, \end{cases} \quad (7b)$$

where K_2 is as defined in Theorem 3 below.

Proof. To find the bias of c_k we write

$$c_k = \tilde{c}_k - (1 + k/n)(\bar{y} - u)^2 + n^{-1}(\bar{y} - u)\left\{\sum_{t=1}^k (y_t - u) + \sum_{t=1}^k (y_{n-t+1} - u)\right\}. \quad (8)$$

The first term on the right side of (8) has expectation $\gamma_k + O(n^{-1})$, the second term is of order $O_p(n^{-\alpha})$ and has expectation $-2\lambda n^{-\alpha}/\{(1-\alpha)(2-\alpha)\} + o(n^{-\alpha})$ by Lemma 2, and the third term is of order $O_p(n^{-1-\alpha/2})$ and is asymptotically negligible. Hence we deduce (6).

To establish (7) we consider the cases $0 < \alpha < 1/2$, $\alpha = 1/2$ and $1/2 < \alpha < 1$ separately.

First suppose $0 < \alpha < 1/2$. The covariance of c_k and c_ℓ is given, apart from terms of order n^{-1} , by Anderson (1971, p. 452, equation (65)). The expression involves terms arising from the variance and kurtosis of $\{a_t\}$. The variance terms are the same as if $\{a_t\}$ were Normally distributed and sum to $2\lambda^2 K_2 n^{-2\alpha}$ asymptotically, as shown in Theorem 3 below. To prove (7a) we must show that the kurtosis terms make an asymptotically negligible contribution to $\text{cov}(c_k, c_\ell)$. These terms are

$$\begin{aligned} & \kappa n^{-2} \sum_{t=1}^n \sum_{u=1}^n \left(\sum_{i=0}^{\infty} \psi_i \psi_{i+k} \psi_{i+u-t} \psi_{i+l+u-t} - n^{-1} \sum_{v=1}^n \sum_{i=0}^{\infty} \psi_i \psi_{i+k} \psi_{i+u-t} \psi_{i+l+v-t} \right. \\ & \quad \left. - n^{-1} \sum_{s=1}^n \sum_{i=0}^{\infty} \psi_i \psi_{i+k} \psi_{i+u-t} \psi_{i+l+k+u-s} + n^{-2} \sum_{s=1}^n \sum_{v=1}^n \sum_{i=0}^{\infty} \psi_i \psi_{i+k+u-s} \psi_{i+u-t} \psi_{i+l+v-t} \right), \end{aligned} \quad (9)$$

where $\psi_j = 0$ for $j < 0$. Now since $\psi_j \sim v_j^{-\alpha}$ as $j \rightarrow \infty$ we have $|\psi_j| < v_j$ for all integers j , where $v_j = 0$ if $j < 0$, $v_0 = C$, $v_j = C j^{-\alpha}$ if $j > 1$, and $C > 0$ is a constant. Note that $\{v_j : j > 0\}$ is a positive decreasing sequence. Thus

$$\begin{aligned}
& |n^{-2} \sum_{t=1}^n \sum_{u=1}^n \sum_{i=0}^{\infty} \psi_i \psi_{i+k} \psi_{i+u-t} \psi_{i+l+u-t}| \leq n^{-2} \sum_{t=1}^n \sum_{u=1}^n \sum_{i=0}^{\infty} \psi_i \psi_{i+k} \psi_{i+u-t} \psi_{i+l+u-t} \\
& \leq n^{-2} \{ n \sum_i \psi_i^2 \psi_{i+k}^2 \psi_{i+l}^2 + \sum_{t < u} \sum_i \psi_i \psi_{i+k} \psi_{i+u-t} \psi_{i+l+u-t} + \sum_{t > u} \sum_i \psi_{i+t-u} \psi_{i+k+t-u} \psi_i \psi_{i+2} \} \\
& \leq n^{-2} \sum_t \sum_u \sum_i \psi_i^2 \psi_{i+|t-u|}^2 \\
& \leq n^{-2} \sum_{t=1}^n \sum_{u=1}^n \left(\sum_{i=0}^{\infty} \psi_i^4 \right)^{1/2} \left(\sum_{i=0}^{\infty} \psi_{i+|t-u|}^4 \right)^{1/2} \text{ by Cauchy's inequality} \\
& = n^{-1} \sum_{i=0}^{\infty} \psi_i^4 + 2n^{-2} \left(\sum_{i=0}^{\infty} \psi_i^4 \right)^{1/2} \sum_{t=1}^n (n-t) \left(\sum_{i=0}^{\infty} \psi_{i+t}^4 \right)^{1/2}.
\end{aligned}$$

Now $\psi_i^4 = O(i^{-4\beta})$, so $\sum_{i=0}^{\infty} \psi_i^4 < \infty$, $g_t^2 \equiv \sum_{i=0}^{\infty} \psi_{i+t}^4 = O(t^{1-4\beta})$ and

$$\sum_{t=1}^n (n-t) g_t = n \sum_t g_t - \sum_t t g_t = O(n^{5/2-2\beta}).$$

Thus the first term of (9) is $O(n^{1/2-2\beta}) = O(n^{-1/2-\alpha}) = o(n^{-2\alpha})$. A similar argument shows that the other terms of (9) are $O(n^{1/2-2\beta})$ also. For example for the second term we write

$$\begin{aligned}
& |n^{-3} \sum_{t=1}^n \sum_{u=1}^n \sum_i \psi_i \psi_{i+k} \psi_{i+u-t} \psi_{i+l+v-t}| \leq n^{-3} \sum_{t=1}^n \sum_{u=1}^n \sum_i \psi_i \psi_{i+k} \psi_{i+u-t} \psi_{i+l+v-t} \\
& \leq n^{-3} \sum_{t=1}^n \sum_{u=1}^n \left(\sum_i \psi_i \psi_{i+k} \psi_{i+u-t}^2 \right)^{1/2} \left(\sum_i \psi_i \psi_{i+k} \psi_{i+l+v-t}^2 \right)^{1/2} \text{ (Cauchy's inequality)} \\
& \leq n^{-3} \sum_{t=1}^n \left(\sum_{u=1}^n h_{t-u} \right)^2
\end{aligned}$$

where $h_t = O(t^{1/4-\beta})$ as $t \rightarrow \infty$, whence an argument like the above shows that this term too is of order $O(n^{1/2-2\beta})$. Thus (9) as a whole is of order $o(n^{-2\alpha})$ and this proves (7a).

Next suppose $\alpha = 1/2$. First we consider $\text{cov}(\tilde{c}_k, \tilde{c}_l)$: we have (cf. Anderson, 1971, p. 451)

$$\begin{aligned} \text{cov}(\tilde{c}_k, \tilde{c}_\ell) &= n^{-2} \sum_{t=1}^{n-k} \sum_{u=1}^{n-\ell} (\gamma_{t-u} \gamma_{t-u+k-\ell} + \gamma_{t-u+k} \gamma_{t-u-\ell} \\ &\quad + \kappa \sum_{i=0}^{\infty} \psi_i \psi_{i+k} \psi_{i+u-t} \psi_{i+\ell+u-t}) . \end{aligned} \quad (10)$$

The terms involving γ_j in (10) yield

$$\begin{aligned} &n^{-2} \sum_{t=1}^{n-k} \sum_{u=1}^{n-\ell} (\gamma_{t-u} \gamma_{t-u+k-\ell} + \gamma_{t-u+k} \gamma_{t-u-\ell}) \\ &\sim n^{-2} \sum_{t=1}^n \sum_{u=1}^n (\gamma_{t-u} \gamma_{t-u+k-\ell} + \gamma_{t-u+k} \gamma_{t-u-\ell}) \text{ as } n \rightarrow \infty \\ &= n^{-1} (\gamma_0 \gamma_{k-\ell} + \gamma_k \gamma_\ell) + n^{-2} \sum_{t \neq u} (\gamma_{t-u} \gamma_{t-u+k-\ell} + \gamma_{t-u+k} \gamma_{t-u-\ell}) \\ &= n^{-1} (\gamma_0 \gamma_{k-\ell} + \gamma_k \gamma_\ell) + 2n^{-2} \sum_{t=1}^n (n-t) \{2\lambda^2 t^{-1} + o(t^{-1})\} \end{aligned} \quad (11)$$

since $\gamma_k \sim \lambda k^{-1/2}$ as $k \rightarrow \infty$. Asymptotically the dominant term in (11) is $4\lambda^2 n^{-1} t^{-1} \sim 4\lambda^2 n^{-1} \log n$; the other terms are asymptotically negligible. The identical argument to that used in the case $0 < \alpha < 1/2$ above shows that the kurtosis term in (10) is $O(n^{-1}) = o(n^{-1} \log n)$, so we have proved that

$$\text{cov}(\tilde{c}_k, \tilde{c}_\ell) \sim 4\lambda^2 n^{-1} \log n \text{ as } n \rightarrow \infty, \quad (12)$$

and hence that \tilde{c}_k is of stochastic order $n^{-1/2}(\log n)^{1/2}$. Now from (8) it follows that $c_k - \tilde{c}_k = o_p(n^{-1/2})$ because by Lemma 2 $(\bar{y} - \mu)^2 = o_p(n^{-1/2})$. Thus replacement of \tilde{c}_k by c_k has an asymptotically negligible effect and (7b) follows from (12).

Finally when $1/2 < \alpha < 1$, the sum of squared autocovariances, $\sum \gamma_k^2$, is convergent. Thus the spectrum of the process is square-integrable and (7c) follows from the central limit theorem of Hannan (1976).

Theorem 3

Let $\{y_t\}$ be a time series satisfying (1)-(3).

(i) If $0 < \alpha < 1/2$ and (4d) holds, let $C_k = n^\alpha(c_k - \gamma_k)$; then as $n \rightarrow \infty$, $C_k - C_\ell \xrightarrow{P} 0$ for $k \neq \ell$ and the common limiting distribution of the C_k has r th cumulant

$$\kappa_r = \lambda^r 2^{r-1} (r-1)! K_r \quad (13)$$

where

$$K_1 = -2/\{(1-\alpha)(2-\alpha)\}, \quad (14)$$

$$K_r = \int_0^1 \dots \int_0^1 g(x_1, x_2) g(x_2, x_3) \dots g(x_{r-1}, x_r) g(x_r, x_1) dx_1 \dots dx_r, \quad r \geq 2, \quad (15)$$

with

$$g(x, y) = |x - y|^{-\alpha} - \{x^{1-\alpha} + (1-x)^{1-\alpha} + y^{1-\alpha} + (1-y)^{1-\alpha}\}/(1-\alpha) + 2/\{(1-\alpha)(2-\alpha)\}. \quad (16)$$

(ii) If $\alpha = 1/2$ and (4d) holds, let $C_k = (n/\log n)^{1/2}(c_k - \gamma_k)$; then as $n \rightarrow \infty$, $C_k - C_\ell \xrightarrow{P} 0$ for $k \neq \ell$ and the common limiting distribution of the C_k is $N(0, 4\lambda^2)$.

(iii) If $1/2 < \alpha < 1$ and (4b) holds, let $C_k = n^{1/2}(c_k - \gamma_k)$; then as $n \rightarrow \infty$ any finite subset of the C_k has a limiting distribution which is multivariate Normal with mean zero and covariances given by (7c).

Remark. Rosenblatt (1979) proved the corresponding result to Theorem 3(i) for the asymptotic distribution of the autocovariances \tilde{c}_k calculated assuming the mean μ to be known. The limiting distribution is the same as that defined by (13)-(16) except that the function $g(x, y)$ is replaced by $\tilde{g}(x, y) = |x - y|^{-\alpha}$. These distributions are further discussed in Section 5.

Proof. (i) We adapt Rosenblatt's (1979) proof of his proposition to the case in which the sample mean is estimated. Write $z_t^{(n)} = y_t - \bar{y}$ so that

$$c_k = n^{-1} \sum_{t=1}^{n-k} z_t^{(n)} z_{t+k}^{(n)}.$$

For fixed n , the vector $z^{(n)} = (z_1^{(n)} \dots z_n^{(n)})^T$ has a degenerate multivariate Normal distribution with mean zero and covariance matrix $\Omega = (w_{st})$, where

$$w_{st} = \gamma_{s-t} - n^{-1} \sum_{u=1}^n \gamma_{s-u} - n^{-1} \sum_{u=1}^n \gamma_{t-u} + n^{-2} \sum_{u=1}^n \sum_{v=1}^n \gamma_{u-v}$$

and rank $\Omega = n - 1$. The joint characteristic function of C_k , $k = 0, 1, \dots, \ell$, can be written as $|I - 2iL^T\Omega L|^{-1/2}$ where $\Omega = LL^T$, L is an $n \times (n - 1)$ matrix of full column rank (Searle, 1971, p. 68), and

$$A = n^{\alpha-1} \sum_{k=0}^{\ell} t_k J^k$$

where

$$J = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \\ & \ddots & & \ddots & \\ & & \ddots & & \\ 0 & & & 1 & \\ & & & & 0 \end{bmatrix}.$$

For $r > 2$, a typical r th-order cross-cumulant of this distribution looks like

$$2^{r-1} (r-1)! n^{\alpha r-r} \sum_{j_1=1}^n \sum_{j_r=1}^n w_{j_1+\alpha_1, j_2} w_{j_2+\alpha_2, j_3} \dots w_{j_{r-1}+\alpha_{r-1}, j_r} w_{j_r+\alpha_r, j_1}$$

where $\alpha_1, \dots, \alpha_r$ take on values $0, 1, \dots, \ell$. As $n \rightarrow \infty$ all these r th order terms have the same limit, this limit being (15) with

$$g(x, y) = |x - y|^{-\alpha} - \int_0^1 |x - u|^{-\alpha} du - \int_0^1 |y - u|^{-\alpha} du + \int_0^1 \int_0^1 |u - v|^{-\alpha} du dv. \quad (17)$$

It is trivial to show that (16) and (17) are equivalent. That $EC_k + \kappa_k$ follows from Theorem 2. To complete the proof we must show that the cumulants (13)-(16) define a unique distribution. We have by Cauchy's inequality for integrals that

$$|\kappa_{2r}| \leq \left(\int_0^1 \dots \int_0^1 g^2(x_1, x_2) g^2(x_3, x_4) \dots g^2(x_{2r-1}, x_{2r}) dx_1 \dots dx_{2r} \right)^{1/2}.$$

$$\left(\int_0^1 \dots \int_0^1 g^2(x_2, x_3) \dots g^2(x_{2r-2}, x_{2r-1}) g^2(x_{2r}, x_1) dx_1 \dots dx_{2r} \right)^{1/2}$$

$$= \kappa_2^r,$$

and similarly that

$$|\kappa_{2r+1}| \leq \kappa_2^{r-1/2} \left(\int_0^1 \int_0^1 \int_0^1 g^2(x_1, x_2) g^2(x_2, x_3) dx_1 dx_2 dx_3 \right).$$

Noting that $g(x, y) = |x - y|^{-\alpha}$ is bounded for $0 < x < 1, 0 < y < 1$ we can use Rosenblatt's (1979) equation (19) to show that for some constant C , $|\kappa_r| \leq CK_2^{r/2}$ for all r . Thus the joint characteristic function of C_0, C_1, \dots, C_ℓ is analytic in a neighbourhood of the origin and the distribution defined by (13)-(16) is unique.

(ii) When $\alpha = 1/2$ the statistics c_k and \tilde{c}_k are asymptotically equivalent, as shown in the proof of Theorem 2, so to prove Theorem 3(ii) it is sufficient to establish the asymptotic joint Normality of $\tilde{c}_k = (n/\log n)^{1/2}(\tilde{c}_k - \gamma_k)$, $k = 0, 1, \dots, \ell$. By following the approach of Rosenblatt (1979, p. 127) we see that a typical r th cross-cumulant of the joint distribution of $\tilde{c}_0, \dots, \tilde{c}_\ell$ looks like

$$u_r = 2^{r-1}(r-1)!(n \log n)^{-r/2} \sum_{j_1=1}^n \sum_{j_r=1}^n \gamma_{j_1+\alpha_1-j_2} \gamma_{j_2+\alpha_2-j_3} \dots \gamma_{j_{r-1}+\alpha_{r-1}-j_r} \gamma_{j_r+\alpha_r-j_1}$$

where $\alpha_1, \dots, \alpha_r$ take on values $0, 1, \dots, \ell$. We will show that for $r > 3$ these cumulants tend to zero as $n \rightarrow \infty$. We take $\alpha_1 = \dots = \alpha_r = 0$ for convenience but our proof is also valid, with minor modifications, when the α_j are not all zero. By Cauchy's inequality we have

$$|u_r| \leq 2^{r-1}(r-1)!(n \log n)^{-r/2} \sum_{j_1=1}^n \dots \sum_{j_r=1}^n \gamma_{j_1-j_2}^2 |\gamma_{j_2-j_3}| |\gamma_{j_3-j_4}| \dots |\gamma_{j_{r-1}-j_r}|. \quad (18)$$

Now $y_j = O(j^{-1/2})$ as $j \rightarrow \infty$, so for $1 \leq j \leq n$

$$\sum_{i=1}^n |y_{i-j}| = y_0 + \sum_{i=1}^{j-1} |y_i| + \sum_{i=1}^{n-j} |y_i| \\ < c'(1 + j^{1/2} + (n-j)^{1/2}) < 3Cn^{1/2} = O(n^{1/2}) \quad (19)$$

for some constant $C > 0$, since $\sum_{i=1}^j i^{-1/2} = O(j^{1/2})$, similarly

$$\sum_{i=1}^n y_{i-j}^2 = y_0^2 + \sum_{i=1}^{j-1} y_i^2 + \sum_{i=1}^{n-j} y_j^2 \\ < c'(1 + \log j + \log(n-j)) < 3C' \log n = O(\log n) \quad (20)$$

for some constant $C' > 0$, since $\sum_{i=1}^j i^{-1} = O(\log j)$. Summing (18) over j_1, \dots, j_r successively and using (19) and (20) we have

$$u_r = (n \log n)^{-r/2} \cdot O(\log n) \cdot O(n^{(r-2)/2}) \cdot n = O((\log n)^{1-r/2})$$

and so $u_r \rightarrow 0$ for $r \geq 3$. Thus the \tilde{C}_k , and by asymptotic equivalence the C_k also, are asymptotically jointly Normal and Theorem 3(iii) follows, the variance of the limiting Normal distribution of C_k being obtained from Theorem 2.

(iii) When $1/2 < \alpha < 1$, $\{y_t\}$ satisfies the conditions of the central limit theorem of Hannan (1976), as remarked in the proof of Theorem 2 above, and Theorem 3(iii) follows in consequence.

4. SAMPLE AUTOCORRELATIONS

The autocorrelations of the time series $\{y_t\}$ are the quantities $\rho_k = \gamma_k/\gamma_0$, and are estimated from an observed series $\{y_t : t = 1, \dots, n\}$ by the sample autocorrelations

$$r_k = \frac{\sum_{t=1}^{n-k} (y_t - \bar{y})(y_{t+k} - \bar{y})}{\sum_{t=1}^n (y_t - \bar{y})^2} = c_k/c_0.$$

As with the sample autocovariances, a number of asymptotically equivalent variants of r_k may be defined. The asymptotic properties of sample autocorrelations are qualitatively similar to those of sample autocovariances. For long-memory time series with autocovariance function (1), the same trichotomy as in Section 3 applies: for $\frac{1}{2} < \alpha < 1$, r_k has the "standard asymptotic behaviour" of asymptotic Normality and variance of order n^{-1} ; for $\alpha = \frac{1}{2}$, r_k is asymptotically Normal but with variance of order $n^{-1}\log n$; and for $0 < \alpha < \frac{1}{2}$, r_k has variance of order $n^{-2\alpha}$ and an asymptotic distribution which is not Normal.

Theorem 4

Let $\{y_t\}$ be a time series satisfying (1)-(3) and (4b). Then the asymptotic bias and covariance of the r_k , $k \geq 1$, are given by

$$E r_k - \rho_k \sim \frac{-2(1 - \rho_k)}{(1 - \alpha)(2 - \alpha)} \cdot \frac{\lambda}{\gamma_0} \cdot n^{-\alpha} \text{ as } n \rightarrow \infty,$$

$$\text{cov}(r_k, r_\ell) \sim \begin{cases} 2(\lambda/\gamma_0)^2(1 - \rho_k)(1 - \rho_\ell)K_2 n^{-2\alpha} & \text{if } 0 < \alpha < \frac{1}{2}, \\ 4(\lambda/\gamma_0)^2(1 - \rho_k)(1 - \rho_\ell)n^{-1}\log n & \text{if } \alpha = \frac{1}{2}, \end{cases} \quad (21a)$$

$$\text{cov}(r_k, r_\ell) \sim \begin{cases} n^{-1} \left\{ \sum_{s=-\infty}^{\infty} (\rho_s \rho_{s+k-\ell} + \rho_s \rho_{s+k+\ell} + 4\rho_k \rho_\ell \rho_s^2 - 2\rho_k \rho_s \rho_{s+\ell} - 2\rho_\ell \rho_s \rho_{s+k}) \right\} & \text{if } \frac{1}{2} < \alpha < 1, \end{cases} \quad (21b)$$

$$\text{cov}(r_k, r_\ell) \sim \begin{cases} n^{-1} \left\{ \sum_{s=-\infty}^{\infty} (\rho_s \rho_{s+k-\ell} + \rho_s \rho_{s+k+\ell} + 4\rho_k \rho_\ell \rho_s^2 - 2\rho_k \rho_s \rho_{s+\ell} - 2\rho_\ell \rho_s \rho_{s+k}) \right\} & \text{if } \frac{1}{2} < \alpha < 1, \end{cases} \quad (21c)$$

where K_2 is as defined in Theorem 3.

Proof. Write $r_k - \rho_k = (c_k - \rho_k c_0)/c_0$. The results then follow from Taylor expansion of the c_k about their expectations γ_k , using Theorem 2 above and the techniques of Fuller (1976, Section 5.4).

Theorem 5

Let $\{y_t\}$ be a time series satisfying (1)-(3).

(i) If $0 < \alpha < \frac{1}{2}$ and (4d) holds, let $R_k = n^\alpha(r_k - \rho_k)/(1 - \rho_k)$; then as $n \rightarrow \infty$, $R_k - R_\ell \xrightarrow{P} 0$ for $k \neq \ell$ and the common limiting distribution of the R_k has r th cumulant $\gamma_0^{-r} \kappa_r$ where κ_r is defined by (13)-(16).

(ii) If $\alpha = \frac{1}{2}$ and (4d) holds, let $R_k = (n/\log n)^{1/2}(r_k - \rho_k)/(1 - \rho_k)$; then as $n \rightarrow \infty$, $R_k - R_\ell \xrightarrow{P} 0$ for $k \neq \ell$ and the common limiting distribution of the R_k is $N(0, 4(\lambda/\gamma_0)^2)$.

(iii) If $\frac{1}{2} < \alpha < 1$ and (4a) holds, let $R_k = n^{1/2}(r_k - \rho_k)$; then as $n \rightarrow \infty$ any finite subset of the R_k , $k \geq 1$, has a limiting distribution which is multivariate Normal with mean zero and covariances given by (21c).

Proof. First suppose that $0 < \alpha < \frac{1}{2}$. Writing $r_k - \rho_k = \{(c_k - \gamma_k) - \rho_k(c_0 - \gamma_0)\}/c_0$ we have

$$R_k = c_0^{-1}(c_k - \rho_k c_0)/(1 - \rho_k) \quad (22)$$

where c_k, c_0 are as defined in Theorem 3(i). From that theorem it follows that $c_k - \rho_k c_0$ has the same limiting distribution as $(1 - \rho_k)c_0$ and this together with the result that $c_0 \rightarrow \gamma_0$ almost surely (Hannan and Heyde, 1972) implies that R_k has a limiting distribution which is identical to that of $\gamma_0^{-1}c_0$ and hence has r th cumulant $\gamma_0^{-r} \kappa_r$ where κ_r is defined by (13)-(16). From (22) we also have

$$R_k - R_\ell = c_0^{-1}\{(c_k - c_0)/(1 - \rho_k) - (c_\ell - c_0)/(1 - \rho_\ell)\}$$

and since $c_k - c_0 \xrightarrow{P} 0$, $c_\ell - c_0 \xrightarrow{P} 0$ and $c_0 \rightarrow \gamma_0$ almost surely this implies that $R_k - R_\ell \xrightarrow{P} 0$. This completes the proof of (i).

The proof of (ii) is almost identical to that of (i): only the limiting distribution of c_0 is different.

To prove (iii) we note that $j^{1/2} w_j^2 = O(j^{1/2-2\beta})$ and that $\frac{1}{2} - 2\beta < -1$. Thus $\gamma j^{1/2-2\beta} < \infty$ and (iii) follows from Theorem 3 of Hannan and Heyde (1972).

5. CUMULANTS OF NON-NORMAL ASYMPTOTIC DISTRIBUTIONS OF SAMPLE AUTOCOVARIANCES AND AUTOCORRELATIONS

The non-Normal asymptotic distribution of sample autocovariances and autocorrelations, derived in Theorems 3(i) and 5(i) above, is similar to the limiting distribution of $n^\alpha(\tilde{c}_k - \gamma_k)$ for a process satisfying (1) with $0 < \alpha < \frac{1}{2}$, obtained by Rosenblatt (1979). Rosenblatt's distribution has r th cumulant

$$\tilde{c}_r = \lambda^r 2^{r-1} (r-1)! I_r \quad (23)$$

where

$$I_1 = 0, \quad (24a)$$

$$I_r = \int_0^1 \dots \int_0^1 |x_1 - x_2|^{-\alpha} |x_2 - x_3|^{-\alpha} \dots |x_{r-1} - x_r|^{-\alpha} |x_r - x_1|^{-\alpha} dx_1 \dots dx_r, \quad r \geq 2. \quad (24b)$$

It is of interest to evaluate some of the lower cumulants of this distribution and of its modified version with cumulants defined by (13)-(16), in order to see how far from Normal these distributions are. The following theorem gives some analytic expressions for the integrals which define these cumulants.

Theorem 6.

Let I_r be defined by (24) above and K_r by (14)-(16) above. Let

$$J_{r-1} = \int_0^1 \dots \int_0^1 |x_1 - x_2|^{-\alpha} |x_2 - x_3|^{-\alpha} \dots |x_{r-1} - x_r|^{-\alpha} dx_1 \dots dx_r, \quad r = 1, 2, \dots,$$

i.e. J_{r-1} is similar to I_r but with the term $|x_r - x_1|^{-\alpha}$ omitted from the integrand. Then

$$I_2 = 1/[(1-\alpha)(1-2\alpha)],$$

$$I_3 = 4\Gamma^2(1-\alpha)/\{(2-3\alpha)\Gamma(3-2\alpha)\}$$

$$I_4 = \frac{6\Gamma^3(1-\alpha)}{(3-4\alpha)\Gamma(4-3\alpha)} + \frac{1}{(1-\alpha)^3(3-4\alpha)} {}_3F_2 \left[\begin{smallmatrix} 1, \alpha, 2-2\alpha \\ 2-\alpha, 3-2\alpha \end{smallmatrix} \right] + -\frac{2\Gamma^2(2-2\alpha)}{(1-\alpha)^3\Gamma(4-4\alpha)} {}_3F_2 \left[\begin{smallmatrix} 1, \alpha, \alpha \\ 2-\alpha, 2-\alpha \end{smallmatrix} \right], \quad (25)$$

$$K_2 = I_2 - 2J_2 + J_1^2, \quad (26)$$

$$K_3 = I_3 - 3J_3 + 3J_2J_1 - J_1^3, \quad (27)$$

$$K_4 = I_4 - 4J_4 + 4J_3J_1 + 2J_2^2 - 4J_2J_1^2 + J_1^4; \quad (28)$$

$$J_1 = 2/\{(1-\alpha)(2-\alpha)\},$$

$$J_2 = 2\Gamma^2(1-\alpha)/\Gamma(4-2\alpha) + 2/\{(1-\alpha)^2(3-2\alpha)\},$$

$$J_3 = \frac{2\Gamma^3(1-\alpha)}{\Gamma(5-3\alpha)} + \frac{4\Gamma^2(1-\alpha)}{(1-\alpha)(4-3\alpha)\Gamma(3-2\alpha)} + \frac{2}{(1-\alpha)^3(3-2\alpha)} {}_3F_2 \left[\begin{smallmatrix} 1, \alpha-1, 3-2\alpha \\ 2-\alpha, 4-2\alpha \end{smallmatrix} \right],$$

$$J_4 = \frac{2\Gamma^4(1-\alpha)}{\Gamma(6-4\alpha)} + \frac{4\Gamma^3(1-\alpha)}{(1-\alpha)(5-4\alpha)\Gamma(4-3\alpha)} + \frac{2\Gamma^2(1-\alpha)}{(1-\alpha)^2(4-3\alpha)\Gamma(3-2\alpha)} {}_3F_2 \left[\begin{smallmatrix} 1, \alpha-1, 4-3\alpha \\ 3-2\alpha, 5-3\alpha \end{smallmatrix} \right]$$

$$+ \left\{ \frac{4}{(1-\alpha)^3(3-2\alpha)(5-4\alpha)} + \frac{12\Gamma(1-\alpha)\Gamma(3-3\alpha)}{(1-\alpha)(3-2\alpha)\Gamma(6-4\alpha)} \right\} {}_3F_2 \left[\begin{smallmatrix} 1, \alpha, 3-2\alpha \\ 2-\alpha, 4-2\alpha \end{smallmatrix} \right]$$

$$+ \frac{4}{(1-\alpha)^3(4-3\alpha)} \int_0^1 t^{1-\alpha} F(1, \alpha, 2-\alpha; t) F(\alpha-1, 4-3\alpha, 5-3\alpha; t) dt$$

$$+ \frac{8\Gamma(1-\alpha)\Gamma(2-2\alpha)}{(1-\alpha)\Gamma(5-3\alpha)} \int_0^1 t^{1-\alpha} F(1, \alpha, 2-\alpha; t) F(\alpha-1, 3-2\alpha, 5-3\alpha; t) dt,$$

here $F(\alpha, \beta; \gamma; x)$ is the hypergeometric function and

$${}_3F_2 \left[\begin{smallmatrix} 1, \alpha, \beta \\ \gamma, \delta \end{smallmatrix} \right] \equiv \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{j=0}^{\infty} \frac{\Gamma(j+\alpha)\Gamma(j+\beta)}{\Gamma(j+\gamma)\Gamma(j+\delta)}$$

is a generalized hypergeometric function of unit argument (Bailey, 1935).

Proof. The proof, which is tedious but not difficult, is not given in detail. To prove (26)-(28) we substitute $g(z, y)$ from (17) in (15); the resulting expressions for $r = 2, 3$ and 4 simplify to (26)-(28) respectively. To evaluate the integrals for I_r and J_r we break the region of integration into sections of the form

$0 < x_{i_1} < \dots < x_{i_r} < 1$ for i_1, \dots, i_r a permutation of $1, \dots, r$, and within each section integrate over x_{i_1}, \dots, x_{i_r} in some convenient order. For each integration we transform the dummy variable so that the range of integration is $(0, 1)$ and use the expressions

$$\int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt = B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta) ,$$

$$\int_0^1 t^{\alpha-1} (1-t)^{\beta-1} (1-xt)^{-\gamma} dt = B(\alpha, \beta) F(\alpha, \gamma; \alpha + \beta; x) ,$$

$$\int_0^1 t^{\alpha-1} (1-t)^{\beta-1} F(\gamma, \delta; \epsilon; t) dt = B(\alpha, \beta) {}_3F_2 \left[\begin{matrix} \alpha, \gamma, \delta \\ \alpha + \beta, \epsilon \end{matrix} \right]$$

(Gradshteyn and Ryzhik, 1980, pp. 284, 286, 849). As an example of the method we prove (25). We have

$$I_4 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 |x_1 - x_2|^{-\alpha} |x_2 - x_3|^{-\alpha} |x_3 - x_4|^{-\alpha} |x_4 - x_1|^{-\alpha} dx_1 dx_2 dx_3 dx_4$$

and using the symmetry of the integrand under the cyclic transformation

$x_1 + x_2 + x_3 + x_4 + x_1$ we can break up the integral as

$$[0,1]^4 = 8 \int_{x_1 > x_2 > x_3 > x_4} + 8 \int_{x_1 > x_2 > x_4 > x_3} + 8 \int_{x_1 > x_3 > x_2 > x_4} . \quad (29)$$

For the first term we perform the successive integrations

$$\int_{x_3}^{x_1} (x_1 - x_2)^{-\alpha} (x_2 - x_3)^{-\alpha} dx_2 = (x_1 - x_3)^{1-2\alpha} B(1-\alpha, 1-\alpha) ,$$

$$\int_{x_4}^{x_1} (x_1 - x_3)^{1-2\alpha} (x_3 - x_4)^{-\alpha} dx_3 = (x_1 - x_4)^{2-3\alpha} B(1-\alpha, 2-2\alpha) ,$$

$$\int_0^1 \int_0^{x_1} (x_1 - x_4)^{2-4\alpha} dx_4 dx_1 = 1/((3-4\alpha)(4-4\alpha)) ,$$

thus the first term of (29) is $8B(1-\alpha, 1-\alpha)B(1-\alpha, 2-2\alpha)/((3-4\alpha)(4-4\alpha))$, which simplifies to the first term of (25). The second term of (29), after the integrations

$$\begin{aligned}
& \int_{x_4}^{x_1} (x_1 - x_2)^{-\alpha} (x_2 - x_3)^{-\alpha} dx_2 \\
&= (x_1 - x_4)^{1-\alpha} (x_4 - x_3)^{-\alpha} (1 - \alpha)^{-1} F(1, \alpha; 2 - \alpha; (x_4 - x_1)/(x_4 - x_3)) \\
&= (x_1 - x_4)^{1-\alpha} (x_1 - x_3)^{-\alpha} (1 - \alpha)^{-1} F(\alpha, 1 - \alpha; 2 - \alpha; (x_1 - x_4)/(x_1 - x_3))
\end{aligned}$$

where we have used 9.131.1 of Gradshteyn and Ryzhik (1980, p. 1043),

$$\begin{aligned}
& \int_{x_3}^{x_1} (x_1 - x_4)^{1-2\alpha} (x_4 - x_3)^{-\alpha} F(\alpha, 1 - \alpha; 2 - \alpha; (x_1 - x_4)/(x_1 - x_3)) dx_4 \\
&= (x_1 - x_3)^{2-3\alpha} \int_0^1 t^{-\alpha} (1 - t)^{1-2\alpha} F(\alpha, 1 - \alpha; 2 - \alpha; 1 - t) dt \\
&= (x_1 - x_3)^{2-3\alpha} B(1 - \alpha, 2 - 2\alpha) {}_3F_2 \left[\begin{matrix} \alpha, 1-\alpha, 2-2\alpha \\ 2-\alpha, 3-3\alpha \end{matrix} \right] ,
\end{aligned}$$

and

$$\int_0^1 \int_0^{x_1} (x_1 - x_3)^{2-4\alpha} dx_3 dx_1 = 1/(3 - 4\alpha)(4 - 4\alpha) ,$$

yields

$$\frac{8B(1-\alpha, 2-2\alpha)}{(1-\alpha)(3-4\alpha)(4-4\alpha)} {}_3F_2 \left[\begin{matrix} \alpha, 1-\alpha, 2-2\alpha \\ 2-\alpha, 3-3\alpha \end{matrix} \right] ,$$

which reduces to the last term of (25) after the application of a transformation of the ${}_3F_2$ function given by Bailey (1935, p. 98). Similarly the last term of (29), after integrations over x_2, x_3, x_4 and x_1 successively, yields the second term on the right side of (25). This completes the proof.

We have used Theorem 6 to evaluate the cumulants up to fourth order of the Rosenblatt and "modified Rosenblatt" distributions, whose cumulants $\tilde{\kappa}_r$ and κ_r are defined by (23)-(24) and (16)-(19) respectively. The results are presented in Table 2 for $\lambda = 1$ and various values of α in the range $0 < \alpha < 1/2$. The values I_2 , I_3 and I_4 have also

been calculated by Mandelbrot and Taqqu (1979) using a less accurate method. It can be seen that the modified distribution is closer to Normal than the original Rosenblatt distribution, and that both distributions approach Normality as $\alpha \rightarrow 1/2$. It is also apparent from Table 2 that the mean of the modified Rosenblatt distribution greatly exceeds the standard deviation in absolute value if $\alpha < 0.3$. Thus for a time series with an autocovariance function (1) and $\alpha < 0.3$, the vast majority of realizations of the series will have a sample autocorrelation function which even for large samples significantly underestimates the true autocorrelation function of the time series. The use of the sample autocorrelation function to identify such processes may therefore be very unreliable.

Table 2. Standardized cumulants of the Rosenblatt and modified Rosenblatt distributions.

Cumulants defined by (23)-(24) and (16)-(19) respectively, with $\lambda = 1$.

$$\text{Skewness} = \kappa_3/\kappa_2^{3/2}, \quad \text{kurtosis} = \kappa_4/\kappa_2^2 - 3.$$

α :	0.02	0.10	0.20	0.30	0.40	0.48
<u>Rosenblatt distribution</u>						
Mean	0.00	0.00	0.00	0.00	0.00	0.00
Std. dev.	1.46	1.67	2.04	2.67	4.08	9.81
Skewness	2.83	2.77	2.55	2.07	1.18	0.17
Kurtosis	11.99	11.66	10.35	7.63	3.39	0.23
<u>Modified Rosenblatt distribution</u>						
Mean	-1.03	-1.17	-1.39	-1.68	-2.08	-2.53
Std. dev.	0.032	0.20	0.55	1.21	2.82	9.12
Skewness	1.37	1.20	0.95	0.67	0.33	0.04
Kurtosis	3.53	2.85	1.94	1.08	0.35	0.02

6. FRACTIONALLY DIFFERENCED ARMA PROCESSES

Recent interest in long-memory time-series models has been particularly stimulated by the family of ARIMA(p,d,q) processes in which the differencing parameter d is permitted to be any real number. These processes have been discussed by Granger (1980), Granger and Joyeux (1980), Hosking (1981, 1984), Geweke and Porter-Hudak (1983) and Li and McLeod (1984). It is therefore of interest to apply our previous results to this class of processes.

A time series $\{y_t\}$ is an ARIMA(p,d,q) process if it can be written as

$$\phi(B)\nabla^d(y_t - \mu) = \theta(B)a_t \quad (30)$$

where μ is the mean of the process, ∇^d is the fractional differencing operator (Hosking, 1981, equation (2.1)), $\phi(B) = 1 - \phi_1 B - \dots - \phi_p B^p$ and $\theta(B) = 1 - \theta_1 B - \dots - \theta_q B^q$ are polynomials of degree p and q respectively in the backward-shift operator B defined by $B y_t = y_{t-1}$, and $\{a_t\}$ is a white-noise process consisting of independent and identically distributed random variates with mean zero and variance σ^2 . An ARIMA(p,d,q) process is stationary if $d < 1/2$ and all the roots of the equation $\phi(z^{-1}) = 0$ lie inside the unit circle $|z| = 1$ (Hosking, 1981). It is straightforward to show, using the approach of Theorem 2 of Hosking (1981), that a stationary ARIMA(p,d,q) process with $d \neq 0$ has an autocovariance function which satisfies

$$\gamma_k \sim \frac{\sigma^2 f_u(0) \Gamma(1 - 2d)}{\Gamma(d) \Gamma(1 - d)} k^{2d-1} \text{ as } k \rightarrow \infty, \quad (31)$$

and an infinite moving-average representation (2) with

$$\psi_j \sim \frac{\{f_u(0)\}^{1/2}}{\Gamma(d)} j^{d-1} \text{ as } j \rightarrow \infty;$$

here

$$f_u(0) = (1 - \theta_1 - \dots - \theta_q)^2 / (1 - \phi_1 - \dots - \phi_p)^2 \quad (32)$$

is the spectral density at frequency zero of the process $\phi(B)u_t = \theta(B)a_t$, the "ARMA part" of the ARIMA(p,d,q) model. We see that for $0 < d < 1/2$ the ARIMA(p,d,q) processes are

stationary long-memory processes whose autocovariance functions have the power-law decay

(1) with exponent $\alpha = 1 - 2d$. Thus Theorems 1-5 above are valid for ARIMA(p,d,q) processes.

Hosking (1981) suggests that the range of values $-\frac{1}{2} < d < \frac{1}{2}$ may be of particular interest when using fractionally differenced ARMA processes for time series modelling. We can without difficulty extend Theorems 1-5 to cover the range $-\frac{1}{2} < d < 0$. Indeed the results stated in Theorems 2-5 for the case $\frac{1}{2} < \alpha < 1$ apply also to the case $\alpha > 1$, or $d < 0$ in (30), since the relevant conditions of Hannan and Heyde (1972) and Hannan (1976) still apply when $\alpha > 1$. The equivalent of Theorem 1 for the case $-\frac{1}{2} < d < \frac{1}{2}$ in (30) is given below.

Theorem 7

Let $\{y_t\}$ be a stationary ARIMA(p,d,q) process (30) with $\frac{1}{2} < d < \frac{1}{2}$. Define $f_u(0)$ as in (32) above.

(i) If $-\frac{1}{2} < d < \frac{1}{2}$, then $n^{1/2-d}(\bar{y} - \mu) \xrightarrow{w} N(0, \omega^2)$ as $n \rightarrow \infty$, where

$$\omega^2 = \frac{\sigma^2 f_u(0) \Gamma(1-2d)}{(1+2d) \Gamma(1+d) \Gamma(1-d)}.$$

(ii) If $d = -\frac{1}{2}$, then $n(\log n)^{-1/2}(\bar{y} - \mu) \xrightarrow{w} N(0, \delta^2)$, where $\delta^2 = 2\pi^{-1}\sigma^2 f_u(0)$.

Proof. (i) For $d = 0$ this is a standard result (Anderson, 1971, Theorem 7.7.8). The results for $0 < d < \frac{1}{2}$ and $-\frac{1}{2} < d < 0$ follow respectively from Theorem 1 and the Remark thereto.

(ii) Now let $d = \frac{1}{2}$ and suppose first that $p = q = 0$. Then $\{y_t\}$ has variance $\gamma_0 = 4\pi^{-1}$ and correlation function $\rho_k = -1/(4k^2 - 1)$ (Hosking, 1981). We write (5) as

$$\text{var } \bar{y} = n^{-2} \gamma_0 \sum_{k=0}^{n-1} \sum_{j=-k}^k \rho_k.$$

By induction we can show that

$$\sum_{j=-k}^k \rho_j = 1/(2k + 1),$$

whence it follows that $\text{var } \bar{y} \sim n^{-2} \gamma_0^{1/2} \log n$ as $n \rightarrow \infty$. It is now straightforward to verify that $\{y_t\}$ satisfies the conditions of Theorem 18.6.5 of Ibragimov and Linnik (1971) and thence that Theorem 7(ii) is valid for an ARIMA(0, $-\frac{1}{2}$, 0) process.

When p and q are not both zero we define the process $\{x_t\}$ by $\nabla^{-1/2} x_t = a_t$, so that $y_t - \mu = \{\phi(B)\}^{-1} \theta(B) x_t$. Now $\{x_t\}$ is an ARIMA(0, $-\frac{1}{2}$, 0) process and, as we have just proved, its mean $\bar{x} = n^{-1}(x_1 + \dots + x_n)$ satisfies $n(\log \bar{x})^{-1/2} \xrightarrow{w} N(0, 2\pi^{-1}\sigma^2)$ as $n \rightarrow \infty$; while the linear filter $\{\phi(B)\}^{-1} \theta(B)$ which transforms x_t into $y_t - \mu$ has a continuous spectrum which takes the value $f_u(0)$ at frequency zero. Applying Theorem 18.6.4 of Ibragimov and Linnik (1971) we obtain $n(\log n)^{-1/2} (\bar{y} - \mu) \xrightarrow{w} N(0, 2\pi^{-1}\sigma^2 f_u(0))$, the required result.

The asymptotic distributions of sample autocovariances and autocorrelations of ARIMA(p, d, q) processes follow directly from Theorems 2-5. The cases $0 < a < \frac{1}{2}$, $a = \frac{1}{2}$ and $\frac{1}{2} < a < 1$ correspond to $\frac{1}{4} < d < \frac{1}{2}$, $d = \frac{1}{4}$ and $0 < d < \frac{1}{4}$ respectively. When $-\frac{1}{2} < d < 0$ the asymptotic distributions are Normal with mean zero and covariances given by (7c) and (21c), i.e. the same as when $0 < d < \frac{1}{4}$. The quantities α , λ and λ/γ_0 occurring in the expressions for asymptotic variances are, from (31) and Hosking (1981),

$$\alpha = 1 - 2d, \quad \lambda = \frac{\sigma^2 f_u(0) \Gamma(1 - 2d)}{\Gamma(d) \Gamma(1 - d)}, \quad \lambda/\gamma_0 = f_u(0) \frac{\Gamma(1 - d)}{\Gamma(d)}.$$

The expressions (7c) and (21c) can also be simplified for certain processes, as we now show.

Lemma 3

Let $\{y_t\}$ be the ARIMA(0, d , q) process $\nabla^d (y_t - \mu) = \theta(B) a_t$ with $d < \frac{1}{4}$ and $Ea_t^4 = \sigma^4(3 + \kappa) < \infty$. Then as $n \rightarrow \infty$

$$\text{cov}(c_k, c_\ell) \sim n^{-1} \sigma^4 (\gamma_{k-\ell}^* + \gamma_{k+\ell}^*) + \kappa \gamma_k \gamma_\ell \quad (33)$$

and

$$\text{cov}(r_k, r_\ell) \sim n^{-1} (\gamma_0^*/\gamma_0)^2 (\rho_{k-\ell}^* + \rho_{k+\ell}^* + 4\rho_k \rho_\ell - 2\rho_k^* \rho_\ell^* - 2\rho_k^* \rho_\ell) , \quad (34)$$

where γ_k^* and ρ_k^* are respectively the lag-k autocovariance and lag-k autocorrelation of the ARIMA(0,2d,q) process $\nabla^{2d}(y_t - \mu) = \theta(B)b_t$ with white-noise variance $Eb_t^2 = 1$, and are given by

$$\gamma_k^* = \sum_{i=-q}^q \sum_{j=-q}^q \gamma_i^u \gamma_j^u \gamma_{k+i-j}^+, \quad \rho_k^* = \gamma_k^*/\gamma_0^*,$$

where

$$\gamma_k^+ = \frac{\Gamma(1-4d)\Gamma(k+2d)}{\Gamma(2d)\Gamma(1-2d)\Gamma(k+1-2d)}, \quad \gamma_k^u = \sum_{i=-q}^{q-k} \theta_i \theta_{i+k}$$

are the lag-k autocovariances of the ARIMA(0,2d,0) process $\nabla^{2d}y_t = b_t$ and the MA(q) process $y_t = \theta(B)b_t$ respectively.

Proof. Let γ_k^x be the lag-k autocovariance of an ARIMA(0,d,0) process with white-noise variance 1. We have (Hosking, 1981)

$$\gamma_k^x = \frac{\Gamma(1-2d)\Gamma(k+d)}{\Gamma(d)\Gamma(1-d)\Gamma(k+1-d)},$$

thus

$$\begin{aligned} \sum_{s=-\infty}^{\infty} \gamma_s^x \gamma_{s+k}^x &= \left\{ \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \right\}^2 \sum_{s=-\infty}^{\infty} \frac{\Gamma(s+d)\Gamma(s+k+d)}{\Gamma(s+1-d)\Gamma(s+k+1-d)} \\ &= \frac{\Gamma(1-2d)\Gamma(k+d)\Gamma(1-k-d)\Gamma(1-4d)}{\Gamma(d)\Gamma(1-d)\Gamma(k+1-2d)\Gamma(1-k-2d)} = \gamma_k^+, \end{aligned} \quad (35)$$

where we have used Dougall's formula (Slater, 1966, p. 180) and some manipulations involving the reflection formula for gamma functions, $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$. When $q = 0$ we have $\gamma_k = \sigma^2 \gamma_k^x$ and substitution of (35) in (7c) and (21c) yields (33) and (34). When $q \neq 0$ we note that

$$\gamma_k = \sigma^2 \sum_{j=-q}^q \gamma_j^u \gamma_{k-j}^x$$

and a similar substitution again yields the results (33) and (34).

7. EXTENSIONS

We consider three possible directions in which the results of Theorems 1-5 may be extended. Writing $y_t = \mu + \sum \psi_j a_{t-j}$ as in (2), we may attempt to weaken our assumptions on: (i) the asymptotic form of ψ_j for large j ; (ii) the distribution of the a_t ; (iii) the dependence structure of the a_t .

As Rosenblatt (1979) remarks in a similar context, Theorems 1, 3(i) and 5(i) can be generalized, without any essential change in the proof, to the case in which $\gamma_k \sim k^{-\alpha} L(k)$ where L is a slowly varying function and the normalization $n^\alpha L(n)$ replaces n^α . Theorems 3(iii) and 5(iii) remain true when $\gamma_k \sim k^{-\alpha} L(k)$, and with the same $n^{1/2}$ normalization. A possibility for further research is to consider the extension of Theorems 3 and 5 to seasonal long-memory processes such as the seasonal ARIMA(0,d,0) process $(1 - B^s)^d y_t = a_t$ where s is an integer and $0 < d < 1/2$.

The assumption of Normality of $\{a_t\}$ in Theorems 3 and 5 for $0 < \alpha < 1/2$ seems unnecessarily strict. If $\{y_t\}$ satisfies (2) and (3) with $Ea_t^{2m} < \infty$ it does not seem difficult to show that, in the notation of Theorem 3, $EC_k^r + \kappa_r \rightarrow n + \infty$ for $r = 1, \dots, m$. This would imply that condition (4d) could be weakened to (4c) without affecting the validity of Theorems 3 and 5. However, since the asymptotic distributions of c_k and r_k when $0 < \alpha < 1/2$ do not involve any higher moments of a_t than the variance, one might expect that these limiting distributions could be obtained under no stronger an assumption on the distribution of a_t than (4a).

The assumption that the a_t in (2) are independent can be relaxed in certain circumstances, notably in Theorems 2-5 when $1/2 < \alpha < 1$ (Hannan and Heyde 1972, Hannan 1976). Similar extensions may be entertained for our other results and their generalizations suggested in this section.

APPENDIX. PROOF OF THEOREM 18.6.5 OF IBRAGIMOV AND LINNIK (1971).

Theorem 18.6.5 of Ibragimov and Linnik (1971, p. 359) is as follows. Let $\{x_j\}$ be a sequence of independent, identically distributed random variables with $Ex_0 = 0$, $Ex_0^2 < \infty$, and let

$$y_j = \sum_{k=-\infty}^{\infty} c_{k-j} x_k$$

where

$$\sum_{k=-\infty}^{\infty} c_k^2 < \infty.$$

If $\sigma_n^2 = E(y_1 + \dots + y_n)^2 \rightarrow \infty$ as $n \rightarrow \infty$, then

$$\Pr\{(y_1 + \dots + y_n)/\sigma_n < z\} \rightarrow (2\pi)^{-1/2} \int_{-\infty}^z \exp(-\frac{1}{2}u^2) du.$$

The first stage of Ibragimov and Linnik's proof is to let $c_{k,n} = c_{k-1} + \dots + c_{k-n}$ so that

$$\sigma_n^2 = \sum_{k=-\infty}^{\infty} c_{k,n}^2,$$

and show that $c_{k,n}/\sigma_n$ tends to zero uniformly in k as $n \rightarrow \infty$. It is this part of the proof which is in error: equations (18.6.14)-(18.6.16) are all incorrect. A correct proof runs as follows.

We have $c_{j,n} = c_{j-1,n} = c_{j-1} - c_{j-n-1}$, so

$$c_{j,n}^2 = (c_{j-1} - c_{j-n-1})^2 + 2(c_{j-1} - c_{j-n-1})c_{j-1,n} + c_{j-1,n}^2. \quad (A.1)$$

Applying (A.1) k times we have

$$\begin{aligned} c_{j,n}^2 &= \sum_{i=0}^{k-1} (c_{j-i-1} - c_{j-i-n-1})^2 + 2 \sum_{i=0}^{k-1} (c_{j-i-1} - c_{j-i-n-1})c_{j-i-1,n} + c_{j-k,n}^2 \\ &\leq \sum_{i=-\infty}^{\infty} (c_i - c_{i-n})^2 + 2 \sum_{i=-\infty}^{\infty} |(c_i - c_{i-n})c_{i,n}| + c_{j-k,n}^2 \end{aligned}$$

$$< 4 \sum_i c_i^2 + 2 \left\{ \sum_i (c_i - c_{i-n})^2 \sum_i c_{i,n}^2 \right\}^{1/2} + c_{j-k,n}^2$$

$$< 4 \sum_i c_i^2 + 8\sigma_n \left(\sum_i c_i^2 \right)^{1/2} + c_{j-k,n}^2$$

where the last two steps are both applications of Cauchy's inequality. Thus

$$\frac{c_{j,n}^2}{\sigma_n^2} < 4\sigma_n^{-2} \sum_i c_i^2 + 8\sigma_n^{-1} \left(\sum_i c_i^2 \right)^{1/2} + c_{j-k,n}^2 / \sigma_n^2. \quad (A.2)$$

In (A.2) we may choose k so that $c_{j-k,n}^2 / \sigma_n^2$ is arbitrarily small, and this yields

$$|c_{j,n}| / \sigma_n < a_n = [8\sigma_n^{-1} \left\{ \left(\sum_{i=-\infty}^{\infty} c_i^2 \right)^{1/2} + \frac{1}{2} \sigma_n^{-1} \sum_{i=-\infty}^{\infty} c_i^2 \right\}]^{1/2}, \quad (A.3)$$

in which $a_n \rightarrow 0$ as $n \rightarrow \infty$. Equations (A.1), (A.2) and (A.3) are the corrected versions of Ibragimov and Linnik's (18.6.15), (18.6.16) and (18.6.14).

Now defining $a_{k,n} = c_{k,n} / \sigma_n$ (not $a_{k,n} = c_{k,n}$ as stated by Ibragimov and Linnik), the remainder of Ibragimov and Linnik's proof may be followed to obtain the final result.

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ABSTRACT (cont.)

short-memory processes, whose autocovariance functions are absolutely summable. For long-memory processes the variances of the sample mean, and of the sample autocovariances and autocorrelations for $0 < \alpha < \frac{1}{2}$, are not of order n^{-1} asymptotically. When $0 < \alpha < \frac{1}{2}$ the asymptotic distribution of the sample autocovariances and autocorrelations is not Normal.

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